

The Polynomial Carathéodory–Fejér Approximation Method for Jordan Regions

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We propose a method for the approximation of analytic functions on Jordan regions that is based on a Carathéodory–Fejér type of economization of the Faber series. The method turns out to be very effective if the boundary of the region is analytic. It often still works when the region degenerates to a Jordan arc. We also derive related lower and upper bounds for the error of the best approximation.

0. Introduction

IN VIEW OF the great success—both in theory and practice—of the *Carathéodory–Fejér method* (or, *CF method*, for short) for polynomial and rational approximation on a disc (Trefethen, 1981a, b) and on a real interval (Darlington, 1970; Gutknecht & Trefethen, 1982; Trefethen & Gutknecht, 1983), it is natural to ask for a generalization of this method to “arbitrary” domains C in the complex plane. Since the two cases mentioned can be viewed as the economization of power series and Chebyshev series, respectively (cf. the very similar methods in Elliott, 1973; Lam, 1972; Trefethen & Gutknecht, 1983, for the interval), it is also natural to attempt an economization of *Faber series* in the general case. The polynomial version of this *Faber–CF method*—as we call it—is proposed and investigated in this paper; its rational analogue will be treated by Ellacott (1983b) and Gutknecht (1983). As a byproduct we obtain a lower and various upper bounds for the error $E_m(F, C)$ of the best minimax approximation of F on C by a polynomial of degree at most m . The lower bound equals the greatest singular value of an infinite Hankel matrix made up of the coefficients in the Faber series of F (which is assumed to converge uniformly); the matrix becomes finite whenever F itself is a polynomial, which one may well assume in practice.

We start in Section 1 with a short description of the polynomial CF method on the disc and a summary of results used later. Section 2 is devoted to Faber polynomials and the Faber transform and culminates in the derivation of the lower bound for $E_m(F, C)$ mentioned. More complicated upper bounds are derived in

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Section 3 by estimating the error of the Faber–CF approximation. Finally, some numerical results are presented in Section 4.

A related idea, namely the economization of a general Chebyshev series, was briefly mentioned in the dissertation of Elliott (1978).

1. The Polynomial CF Method on the Disc

Let \mathcal{P}_m be the space of complex polynomials of degree m , and denote by $\tilde{\mathcal{P}}_m$ its extension to functions q that are sums of a polynomial $\tilde{p}^+ \in \mathcal{P}_m$ and of a function \tilde{p}^- that is analytic and bounded outside the unit circle S ; i.e. $\tilde{p} \in \tilde{\mathcal{P}}_m$ iff

$$\tilde{p}(w) = \sum_{k=-\infty}^m c_k w^k \quad (|w| > 1). \quad (1.1)$$

On the basis of the classical Carathéodory–Fejér theorem the best approximation \tilde{p} (with respect to the supremum norm $\|\cdot\|$ on S) of a polynomial $f_M \in \mathcal{P}_M$ of (possibly high) degree M out of $\tilde{\mathcal{P}}_m$ ($m < M$) can be computed by solving a singular value problem. In general this costs only a fraction of the numerical computation of the best approximation p^* out of \mathcal{P}_m by currently known methods. Moreover, $\tilde{p} - p^*$ often turns out to be extremely small. This motivates the CF method (Trefethen, 1982):

Given $f \in A(\bar{D})$, i.e. analytic in the unit disc D and continuous in \bar{D} , there exists $f_M \in \mathcal{P}_M$ such that $\|f - f_M\|$ is negligible; we determine the best approximation \tilde{p} to f_M out of $\tilde{\mathcal{P}}_m$ and take $\tilde{p}^+ \in \mathcal{P}_m$ as approximation for f (thus deleting \tilde{p}^-).

As usual we will assume here that the given function f is even analytic in a region containing \bar{D} and that the M th partial sum of the Maclaurin series of f is chosen for f_M :

$$f_M(w) := \sum_{k=0}^M a_k w^k. \quad (1.2)$$

As can be derived from the Carathéodory–Fejér theorem (Trefethen, 1981a), the best approximation \tilde{p} to f_M out of $\tilde{\mathcal{P}}_m$ is unique and its error function

$$q := f_M - \tilde{p}$$

is a scalar multiple of a finite Blaschke product and lies in $\tilde{\mathcal{P}}_M$:

$$q(w) = \sigma w^M \frac{\bar{u}_0 + \dots + \bar{u}_K w^K}{u_K + \dots + u_0 w^K} = \sum_{j=-\infty}^M b_j w^j. \quad (1.3)$$

Here $K := M - m - 1$, $\sigma = \sigma_m^{(M)}$ is the largest singular value of the Hankel matrix

$$A := \begin{pmatrix} a_{m+1} & a_{m+2} & \dots & a_M \\ a_{m+2} & & & \\ \vdots & & & \\ a_M & & & 0 \end{pmatrix}, \quad (1.4)$$

and $u = (u_0, \dots, u_K)^T$ is a corresponding right singular vector satisfying $Au = \sigma \bar{u}$,

which we can choose such that $u_0 \neq 0$. The Laurent coefficients b_k of q satisfy

$$b_j = a_j, \quad m+1 \leq j \leq M, \quad (1.5a)$$

$$b_j = \frac{-1}{u_0} (b_{j+1}u_1 + \dots + b_{j+K}u_K), \quad j \leq m. \quad (1.5b)$$

Finally, the CF approximation $p^{cf} \in \mathcal{P}_m$ is defined by

$$p^{cf} := \bar{p} + q^- = f_M - q^+, \quad (1.6)$$

where

$$q^-(w) := \sum_{j=-\infty}^{-1} b_j w^j, \quad q^+(w) := \sum_{j=0}^M b_j w^j.$$

Trefethen's (1981a) theoretical results require

$$|a_{m+1+j}| \leq |a_{m+1}| \cdot \rho^j, \quad j = 1, \dots, K, \quad (1.7)$$

with $\rho \leq \frac{1}{36}$ or even $\rho \leq 1/(4\sqrt{m+1})$. In particular, he concludes:

- (i) $f - p^{cf}$ and $f - p^*$ have winding number $m+1$ on S (if ρ is small enough),
- (ii) $\|f - p\| - \min_{w \in S} |f(w) - p(w)| = O(\rho^{m+2})\|f - p\|$ both for $p = p^{cf}$ and $p = p^*$ as $\rho \rightarrow 0$,
- (iii) $\|p^{cf} - p^*\| = O(\rho^{m+2})\|f - p^*\|$ as $\rho \rightarrow 0$.

In other words, the error curve of both the best and the CF approximation are nearly circular up to the relative order $O(\rho^{m+2})$ and the two approximations are equal up to this high relative order.

To achieve this ultimate accuracy of the CF method it is necessary to choose $M \geq 2m+2$; the matrix A is then of order $K+1 = m+2$. However, in practical computations with a computer of fixed word length there is no point in choosing K so big if the last few coefficients a_k in A are negligible. For example, if in our original problem

$$\lim_{j \rightarrow \infty} \left| \frac{a_{j+1}}{a_j} \right| = \rho_1 \quad (< 1), \quad (1.8)$$

one will choose K ultimately such that ρ_1^{K+1} is of roundoff level if added to 1. More generally, the same choice is ultimately appropriate whenever f has radius of convergence $1/\rho_1$ at $w = 0$. Note that the singular-value problem is then of fixed size. But of course the total relative error is then at most of order $O(\rho_1^{K+1})$ since the contribution of $f - f_M$ is of this order. (Compare, for instance, that for entire functions, the choice $K = m+1$ leads to $O(\rho_2^{m+2})$ as $m \rightarrow \infty$, with arbitrary $\rho_2 > 0$, cf. Trefethen, 1981a, Thm. 11.)

The error analysis of the CF method is based on an estimate of $\|q^-\|$ since this is a bound for the deviation from circularity. Trefethen (1981a, Lemma 8) obtained

$$\|q^-\| + \|f - f_M\| < \frac{1}{2}(18\rho)^{m+2}|a_{m+1}| \quad \text{if } \rho < \frac{1}{36}.$$

A better bound can be extracted from Hollenhorst's thorough investigations of the error of a real version of polynomial CF approximation (Hollenhorst, 1976, §2); we state it here for later use:

THEOREM 1.1 (Hollenhorst, 1976, p. 59): Let $K \leq m+1$ and let (1.7) be satisfied with $\rho < \frac{1}{8}(\sqrt{13}-1) \doteq 0.43426$. Then all poles of q have modulus at most

$$\xi := \rho \frac{2-3\rho^2}{1-2\rho^2}, \quad (1.9)$$

and

$$\|q^-\| \leq \frac{|a_{m+1}| \xi^{m+1} \rho}{1-2\rho-2\rho^2+3\rho^3}. \quad (1.10)$$

The error analysis becomes particularly simple if K is fixed and

$$\lim_{j \rightarrow \infty} \frac{a_{j+1}}{a_j} = a \quad (0 < |a| < 1)$$

exists, since this case can be treated as the limit of the model problem $a_j = a^j$, see Ellacott & Gutknecht (1983). Here, for any a with $0 < |a| < 1$, there exists $R < 1$ such that $\|q^-\|/|a_{m+1}| = O(R^{-m})$ as $m \rightarrow \infty$ (K fixed).

In two of our estimates it will be sufficient to use the following simple error bound based on Cauchy's coefficient estimate. Admittedly it passes over one of the difficulties by assuming that the poles of q are uniformly bounded away from S .

LEMMA 1.1 Let $\xi \in (0, 1)$ be fixed, and assume all poles of q have modulus at most ξ . Then for any $R \in (\xi, 1)$

$$|b_j| \leq \sigma \mu^K R^{M-j} \quad (j \leq M), \quad (1.11)$$

$$\|q^-\| \leq \sigma \mu^K R^{M+1}/(1-R), \quad (1.12)$$

where $\mu := (1+\xi R)/(R-\xi)$.

Proof. If ζ_1, \dots, ζ_K are the poles of the Blaschke product $w^{-M}q(w)/\sigma$, we get for $|w| = R$

$$\left| \sum_{j=-\infty}^M b_j w^{j-M} \right| = |w^{-M}q(w)| = \sigma \prod_{j=1}^K \left| \frac{\zeta_j w - 1}{w - \zeta_j} \right| \leq \sigma \frac{(1+\xi R)^K}{(R-\xi)^K} = \sigma \mu^K.$$

Hence, by Cauchy's estimate, we obtain the bound in (1.11) for $|b_j|$, and (1.12) follows by summing up all these bounds with $j < 0$. ■

A useful yet fairly trivial set of inequalities following from the optimality of \tilde{p} and from

$$\sigma^2 = \|q\|^2 = \|q\|_2^2 := (2\pi)^{-1} \int_{|w|=1} |q(w)|^2 |dw| = \sum |b_j|^2$$

is

$$\begin{aligned} \|f_M - p^{cf}\| &\geq \|f_M - p^*\| = E_m(f_M, D) \geq \|f_M - \tilde{p}\| \\ &= \|q\| = \sigma = \left[\sum_{j=-\infty}^M |b_j|^2 \right]^{\frac{1}{2}} \geq \max_{j \leq M} |b_j| \geq \max_{m < j \leq M} |a_j|. \end{aligned} \quad (1.13)$$

Trefethen (1981a) showed also that $\|f_M - p^{cf}\|$ and *a fortiori* $E_m(f_M, D)$ and σ tend to $|a_{m+1}|$ if $\rho \rightarrow 0$. (The result for E_m was well known for a long time.) Recently, Henrici (1983) using also the CF method derived an improved estimate for $E_m(f, D)$.

2. Faber Polynomials and Bounds for the Best Polynomial Approximation

Let $C \subset \mathbb{C}$ be the closure of a Jordan region whose boundary Γ is rectifiable and of bounded rotation V . (Bounded rotation is defined as follows (Radon, 1919, p. 1125): consider the function $\gamma: s \mapsto \gamma(s)$ relating the parameter s of a point on the curve with the angle between the tangent at this point and the x -axis; γ is well defined for almost all s ; if γ can be defined at the remaining points such that γ becomes a function of bounded variation, Γ is said to have bounded rotation $V := \int |d\gamma|$.) Let $\|\cdot\|$ now denote the supremum norm either on Γ in the z -plane or on S in the w -plane. Let ϕ ,

$$\phi(z) = dz + d_0 + \frac{d_1}{z} + \dots \quad (z \in \text{ext } \Gamma),$$

be the conformal mapping normalized by $d > 0$ of $\text{ext } \Gamma$ onto $\{w: |w| > 1\}$, and let ψ be its inverse. The n th Faber polynomial $\phi_n \in \mathcal{P}_n$ is defined as the polynomial part of $[\phi(z)]^n$:

$$\phi_n(z) := [\phi(z)]^n + O\left(\frac{1}{z}\right) \quad \text{as } z \rightarrow \infty. \quad (2.1)$$

With this normalization, ϕ_n has leading coefficient d^n . Transplanted to the w -plane ϕ_n satisfies

$$\phi_n(\psi(w)) = w^n + \sum_{k=1}^{\infty} nc_{nk} w^{-k} \quad (|w| > 1) \quad (2.2)$$

(where $c_{nk} = c_{kn}$ according to Grunsky's Law of Symmetry). For the basics on Faber polynomials see, e.g. Markushevich (1967), Smirnov & Lebedev (1968), Curtiss (1971), Gaier (1980).

Let $A(C)$ denote the set of functions continuous on C and analytic in the interior of C . The Faber transform T defined on $A(\bar{D})$ by

$$T: w^n \mapsto \phi_n \quad (n = 0, 1, \dots)$$

and by linear extension is (under our assumption) a bounded linear operator from $A(\bar{D})$ into $A(C)$ with

$$\|T\| \leq 1 + \frac{V}{\pi}. \quad (2.3)$$

This follows easily from Pommerenke's formula (Pommerenke, 1965)

$$\phi_n(\psi(e^{i\theta})) = \frac{1}{\pi} \int_0^{2\pi} e^{int} d_t \arg(\psi(e^{it}) - \psi(e^{i\theta})), \quad n = 1, 2, \dots, \quad (2.4)$$

(for $n = 0$ the right side must be halved) and the inequality

$$\int_0^{2\pi} |d_t \arg(\psi(e^{it}) - \psi(e^{i\theta}))| \leq V \quad (2.5)$$

obtained by taking the limit $w \rightarrow e^{i\theta}$ in Radon's formula (Radon, 1919, p. 1133). (The

argument has to be chosen such that it is continuous for $t \neq \theta$ and has a jump equal to the exterior angle of Γ at $t = \theta$.) The relations (2.4) and (2.5) yield also

LEMMA 2.1 *Let $f \in A(\bar{D})$. Then*

$$\|Tf + f(0)\| \leq \frac{V}{\pi} \|f\|. \quad (2.6)$$

Proof. Let $f(w) = \sum a_n w^n$ ($|w| < 1$). Then for $z = \psi(e^{i\omega})$ one gets according to (2.4) and (2.5)

$$\begin{aligned} |(Tf)(z) + f(0)| &= \left| 2a_0 \phi_0(z) + \sum_{n=1}^{\infty} a_n \phi_n(z) \right| \\ &= \frac{1}{\pi} \left| \int \sum_{n=0}^{\infty} a_n e^{int} d_t \arg(\psi(e^{it}) - \psi(e^{i\omega})) \right| \\ &\leq \frac{V}{\pi} \|f\|. \quad \blacksquare \end{aligned}$$

If $F = Tf$ for some $f \in A(\bar{D})$ and if

$$P(z) := \sum_{k=0}^m c_k \phi_k(z) \quad (2.7)$$

is any polynomial approximation of F , the error $\|F - P\|$ on C can be estimated in terms of the error $\|f - T^{-1}P\|$ on S :

$$\|F - P\| \leq \|T\| \|f - p\|, \quad (2.8)$$

where

$$p(w) := T^{-1}P = \sum_{k=0}^m c_k w^k. \quad (2.9)$$

This trivial remark is of practical importance since the evaluation of polynomials (of possibly high degree) at many equidistant points on S can be done very effectively by the fast Fourier transform. Moreover, if any upper bound for the best approximation error $E_m(f, S)$ of f on S is known, (2.8) gives rise to a bound for the best approximation error $E_m(F, C)$ of f on C :

$$E_m(F, C) \leq \|T\| E_m(f, S).$$

Here and in (2.9) we could insert the bound for $\|T\|$ from (2.3). However, on the basis of Lemma 2.1 improved estimates are obtained immediately (note that $Tf \in \mathcal{P}_m$ implies $Tf + f(0) \in \mathcal{P}_m$):

THEOREM 2.1 (i) *Let $f \in A(\bar{D})$, $F = Tf$, $p, P \in \mathcal{P}_m$, $P = T(p)$. Then*

$$\|F - P\| \leq \frac{V}{\pi} \|f - p\| + |f(0) - p(0)|. \quad (2.10)$$

$$(ii) \quad E_m(F, C) \leq \frac{V}{\pi} E_m(f, S).$$

It is evident from (2.8) or (2.10) that the uniform convergence on S of the Maclaurin series $\sum a_k w^k$ of f implies the uniform convergence on C of the Faber

series

$$\sum_{k=0}^{\infty} a_k \phi_k(z) \quad (2.11)$$

(which has the same coefficients a_k) of $F = Tf$ to F . For non-trivial results on the uniform convergence of Faber series see Kővari & Pommerenke (1967).

Note that the term $|f(0) - p(0)|$ in (2.10) vanishes for many reasonable choices of P , in particular if P is the m th partial sum of the Faber series of F or one of the usual smooth versions of it (Gaier, 1980, pp. 54–57).

In order to obtain from part (ii) of Theorem 2.1 a rigorous upper bound for $E_m(F, C)$ one can for example replace $E_m(f, S)$ there by an estimate for the error of a CF approximant to f . According to (1.6),

$$f - p^f = f - f_M + q^+ = f - f_M + q - q^-,$$

hence

$$E_m(F, C) \leq \frac{V}{\pi} [\sigma + \|f - f_M\| + \|q^-\|], \quad (2.12)$$

where one can further apply the bound (1.10) for $\|q^-\|$ if the assumptions of Theorem 1.1 hold. In practice, $\|f - f_M\|$ and $\|q^-\|$ are negligible for sufficiently smooth functions since the CF method on the disc is then known to work extremely well.

Here we should note that Faber polynomials and Faber series can be defined under different conditions. For example, it suffices that $C \subset \mathbb{C}$ be a compact continuum (containing more than one point) whose complement C^c with respect to the extended plane is simply connected and that F be analytic on C (i.e. at each point of C), see, e.g. Markushevich (1967), Smirnov & Lebedev (1968). Then ψ can be defined almost everywhere on S as a bounded integrable function, and the coefficients a_k in the Faber series (2.12) of F are given by

$$a_k := \frac{1}{2\pi i} \int_S F(\psi(z)) z^{-k-1} dz, \quad k = 0, 1, \dots \quad (2.13)$$

There is one case to which the foregoing treatment can be generalized and which is very useful in practice. Let the complement C^c of C (with respect to the extended plane \mathbb{C}) be a simply connected region (in \mathbb{C}) that is of bounded boundary rotation as defined by Paatero (1931, 1933). Then ψ is still continuous on D^c and the variation of the boundary Γ of C can be defined as for Jordan regions (Paatero, 1931). Moreover, it is easy to check that Pommerenke's proof (Pommerenke, 1964) of (2.4) remains valid, and it is clear that (2.5) is still true. Consequently, Lemma 2.1 and Theorem 2.1 hold also in this case. For example, C may be a Jordan arc; then

$$V := 2\pi + 2V', \quad (2.14)$$

where V' is the variation of the tangent angle along C , while the term 2π is due to the endpoints.

The projection $\Phi_m: F \mapsto F_m$ that associates with a function F having a uniformly convergent Faber series (2.11) its m th partial sum

$$F_m(z) := \sum_{k=0}^m a_k \phi_k(z) \quad (2.15)$$

we call the *Faber projection*. Under our assumptions the following bound holds for its norm:

$$\|\Phi_m\| \leq \frac{V}{\pi} \left(\frac{4}{\pi^2} \ln m + B \right), \quad m = 1, 2, \dots, \quad (2.16)$$

where $B = 1.733 \dots$ is a certain absolute constant, see Ellacott (1983a, Th. 2.3). Since $F - F_m = F - P - \Phi_m(F - P)$ for any $P \in \mathcal{P}_m$,

$$\|F - F_m\| \leq (1 + \|\Phi_m\|) E_m(F, C), \quad (2.17)$$

so that (2.16) implies an *a priori* bound on how good the truncated Faber series must be. However, in practice this estimate is very conservative for well-behaved functions, or, in other words, the truncated Faber series gives a much better approximation than (2.17) would suggest. A much sharper *a posteriori* lower bound is given in the following theorem, which is based on the CF theorem for the disc and the Faber transform:

THEOREM 2.2 (i) Let F_M be given by (2.15) and let $0 \leq m < M$. Then $E_m(F_M, C) \geq \sigma$, where $\sigma = \sigma_m^{(M)}$ is the largest singular value of the Hankel matrix (1.4).

(ii) Let $F = T(f)$, where $f \in A(\bar{D})$ has a uniformly convergent Maclaurin series in \bar{D} . Then

$$E_m(F, C) \geq \lim_{M \rightarrow \infty} \sigma_m^{(M)} = \sigma_m^{(\infty)}, \quad (2.18)$$

where $\sigma_m^{(\infty)}$ is the largest singular value of the semi-infinite Hankel matrix $(a_{k+l+m+1})_{k,l=0}^{\infty}$.

Proof. (i) $E_m(F_M, C) = \min \|c_0 + c_1 \phi_1 + \dots + c_M \phi_M\|$, where the minimum is taken over c_0, \dots, c_m , while $c_j = a_j$ for $j = m+1, \dots, M$. But for any choice of c_0, \dots, c_m

$$\begin{aligned} \sum_{j=0}^M c_j \phi_j(z) &= \sum_{j=0}^m c_j \phi_j(z) + \sum_{j=m+1}^M c_j [\phi_j(z) - \phi_j^J(z)] \\ &= \sum_{j=0}^m c_j w^j + \sum_{j=m+1}^M c_j [\phi_j(\psi(w)) - w^j]. \end{aligned}$$

According to (2.2) the last sum can be expanded purely in terms of negative powers of w for $|w| > 1$ and is bounded for $|w| \geq 1$. Hence, we may write

$$\sum_{j=0}^M c_j \phi_j(z) = \sum_{j=m+1}^M c_j w^j + \tilde{p}(w) \quad (2.19)$$

with $\tilde{p} \in \mathcal{P}_m$ and it follows from the CF theory, cf. Section 1, that the norm of (2.19) is not smaller than $\sigma_m^{(M)}$.

(ii) Since $\sigma_m^{(M)}$ and $\sigma_m^{(\infty)}$ are the errors of the best approximations of f_M and f , respectively, out of \mathcal{P}_m (cf. Adamjan *et al.*, 1971), the assumed uniform convergence of $\{f_M\}$ implies convergence of $\sigma_m^{(M)}$ to $\sigma_m^{(\infty)}$. On the other hand, in view of (2.8), the

Faber series of F converges also uniformly, and hence $E_m(F_M, C)$ converges to $E_m(F, C)$. Finally, the inequality in (2.18) follows from part (i) of this theorem. ■

For approximation on the unit interval $I := [-1, 1]$, Gutknecht & Trefethen (1982) give theoretical and experimental evidence that for sufficiently smooth functions F the truncation error in their corresponding version of the CF method is very small and that the error of the best approximation is close to 2σ (in our notation, when the different scaling of coefficients is allowed for). They give also an example showing that 2σ is in general not a lower bound for $E_m(F, I)$. However, σ is a lower bound according to the previous theorem. Note that subject to the reservations about the applicability of the CF method on the disc, Theorem 2.1 (ii) implies that $E_m(F, C)/\sigma$ cannot be expected to be very much greater than 2 for any convex domain C .

3. The Faber-CF Approximation and Improved Upper Bounds

If we impose further conditions on the boundary Γ and on the function F , upper bounds on $E_m(F, C)$ that are at least asymptotically sharper than the one in (2.12) may be obtained by directly estimating the error of what we call the *Faber-CF approximation* $P^{cf} \in \mathcal{P}_m$ namely the Faber transform $P^{cf} := T p^{cf}$ of the CF approximation to $f_M = T^{-1}F_M$ (with suitable $M > m$).

THEOREM 3.1 *Let Γ be an analytic Jordan curve such that $\psi: \{w: |w| > 1\} \rightarrow \text{ext } \Gamma$ can be extended conformally to $\text{ext } S_r$ where $S_r := \{w: |w| = r\}$, $r < 1$. Let $M = K + m + 1$ with fixed $K \geq 0$, let F and F_M be given by (2.11) and (2.15), respectively, and let $P^{cf} = T p^{cf}$ where p^{cf} is the CF approximation of $T^{-1}F_M$ on \bar{D} . Assume that the poles of the Blaschke product in (1.3) lie in $\bar{D}_\xi := \{w: |w| \leq \xi\}$, $\xi < 1$ (independent of m). Then for each R with $1 > R > \max\{\xi, r\}$*

$$\sigma_m^{(M)} \leq E_m(F_M, C) \leq \|F_M - P_{cf}\| \leq \sigma_m^{(M)}[1 + O(R^m)] \quad (3.1)$$

as $m \rightarrow \infty$. In particular, under the assumptions of Theorem 1.1, ξ can be taken from (1.9), and if r is chosen such that ψ has a homeomorphic extension to $S_r \cup \text{ext } S_r$ and $\Gamma_r := \psi(S_r)$ is a rectifiable Jordan curve of length l_r , then for any M with $m < M \leq 2m + 2$

$$\begin{aligned} \|F_M - P^{cf}\| \leq \sigma_m^{(M)} + \frac{|a_{m+1}| \xi^{m+1} \rho}{1 - 2\rho - 2\rho^2 + 3\rho^3} + \\ \frac{l_r |a_{m+1}|}{2\pi\delta_r} \left\{ \frac{\rho}{1 - 2\rho^2} \frac{\xi^{m+1} - r^{m+1}}{\xi - r} + r^{m+1} \frac{1 - (\rho r)^{M-m-1}}{1 - \rho r} \right\} \end{aligned} \quad (3.2)$$

where δ_r is the distance between Γ_r and Γ .

Proof. We prove (3.2) first, starting from

$$\begin{aligned} F_M(z) - P^{cf}(z) &= T(f_M - p^{cf})(z) = T(q^+)(z) \\ &= \sum_{j=0}^M b_j \phi_j(z) \\ &= \sum_{j=0}^M b_j (\phi_j - \phi^j)(z) + q^+(\phi(z)). \end{aligned} \quad (3.3)$$

Now $\|q^+\| \leq \sigma_m^{(M)} + \|q^-\|$, and the bound (1.10) can be used for $\|q^-\|$. On the other hand,

$$|\phi_j(z) - [\phi(z)]^j| \leq \frac{l_r r^j}{2\pi\delta_r} \quad (j \geq 0, \quad z \in \Gamma_r \cup \text{ext } \Gamma_r) \quad (3.4)$$

as can be seen from a lemma (in Markushevich, 1967, p. 107), and

$$|b_j| \leq \frac{|a_{m+1}| \rho \zeta^{m-j}}{1-2\rho^2} \quad (\forall j \leq m) \quad (3.5)$$

according to Hollenhorst (1976, pp. 64–65), while (1.7) holds for $m < j \leq M$ by assumption. Consequently,

$$\left| \sum_{j=0}^M b_j (\phi_j - \phi^j)(z) \right| \leq \frac{l_r |a_{m+1}|}{2\pi\delta_r} \left\{ \frac{\rho \zeta^m}{1-2\rho^2} \frac{1-(r/\zeta)^{m+1}}{1-r/\zeta} + r^{m+1} \frac{1-(\rho r)^K}{1-\rho r} \right\}. \quad (3.6)$$

Hence, $\|F_M - P^{\zeta f}\|$ is not bigger than the bound in (3.2).

The right-hand side inequality in (3.1) is proved the same way, but r is replaced by r' with $r < r' < R$, so that (3.4) still holds for r' , and (3.5) is replaced by (1.11). On one hand, (1.11) leads to (1.12), on the other hand, by combining it with (3.4) we obtain

$$\left| \sum_{j=0}^M b_j (\phi_j - \phi^j)(z) \right| = O \left(\sigma_m^{(M)} R^M \sum_{j=0}^M \left(\frac{r'}{R} \right)^j \right) = O(\sigma_m^{(M)} R^M). \quad (3.7)$$

Thus,

$$\|F_M - P^{\zeta f}\| = \sigma_m^{(M)} [1 + O(R^M)] = \sigma_m^{(M)} [1 + O(R^m)]. \quad \blacksquare$$

The other case we consider is when C is convex (but not an interval), when we may make use of an idea due to Pommerenke (1964).

THEOREM 3.2 *Let C be convex and let $\alpha\pi$ ($1 \leq \alpha < 2$) be the largest outer angle of its boundary Γ . Let $M = K + m + 1$ with fixed $K \geq 0$, and let F_M be given by (2.15), where $\{a_k\}_{k=0}^\infty$ is a given sequence with the property that the poles of the Blaschke product appearing in (1.3) lie in \bar{D}_ξ with $\xi < 1$ being independent of M . Then*

$$\limsup_{\substack{m, M \rightarrow \infty \\ M-m \text{ fixed} \\ \sigma_m^{(M)} \neq 0}} \frac{E_m(F_M, C)}{\sigma_m^{(M)}} \leq \alpha. \quad (3.8)$$

Proof. Let $v(t, \theta) := \arg(\psi(e^{it}) - \psi(e^{i\theta}))$. According to the proof of Theorem 3 of Pommerenke (1964, p. 204) there exist for any given $\varepsilon > 0$ an integer $J > 1$ and $\delta > 0$ such that for all θ

$$\left| \int_{\theta+\delta}^{2\pi+\theta-\delta} e^{ijt} d_t v(t, \theta) \right| < \varepsilon \quad (\forall j > J), \quad (3.9)$$

$$\int_{\theta-\delta}^{\theta+\delta} |d_t v(t, \theta)| = \int_{\theta-\delta}^{\theta+\delta} d_t v(t, \theta) < \pi(\alpha + \varepsilon). \quad (3.10)$$

The argument in the definition of v is assumed to be chosen such that $v(t, \theta)$ is an

increasing function with a jump of at most $\alpha\pi$ at $t = \theta$ but elsewhere continuous in $[\theta - \delta, 2\pi + \theta - \delta]$ (for fixed θ).

In view of (3.3) and (2.4) we have for $z = \psi(e^{i\theta})$ and $M > J$

$$\begin{aligned} F_M(z) - P^{\mathcal{F}}(z) &= \sum_{j=0}^M b_j \phi_j(z) \\ &= \sum_{j=0}^J b_j \phi_j(z) + \frac{1}{\pi} \sum_{j=J+1}^M b_j \int_0^{2\pi} e^{ijt} d_t v(t, \theta). \end{aligned}$$

From (1.11) we can conclude that the first sum is $O(\sigma_m^{(M)} R^{M-J})$ (uniformly in m) for any $R \in (\xi, 1)$. The integral in the second sum is split in order to apply (3.9) and (3.10); using also (1.3) and (1.11) we get:

$$\begin{aligned} &\left| \int_0^{2\pi} \sum_{j=J+1}^M b_j e^{ijt} d_t v(t, \theta) \right| \\ &\leq \sum_{j=J+1}^M |b_j| \left| \int_{\theta+\delta}^{2\pi+\theta-\delta} e^{ijt} d_t v(t, \theta) \right| + \\ &\quad \left[\max_j \left| \sum_{j=-\infty}^M b_j e^{ijt} \right| + \sum_{j=-\infty}^J |b_j| \right] \int_{\theta-\delta}^{\theta+\delta} |d_t v(t, \theta)| \\ &\leq O(\varepsilon \sigma_m^{(M)}) + \sigma_m^{(M)} \pi(\alpha + \tfrac{1}{2}\varepsilon) + O(\sigma_m^{(M)} R^{M-J}). \end{aligned}$$

Since J is fixed, $R^{M-J} \rightarrow 0$, and since ε was arbitrary, (3.8) follows. ■

4. Numerical Experiments with the Polynomial Faber-CF Method

The Faber-CF approximation $P^{\mathcal{F}}$ defined in Section 3 can be computed easily: Given F with uniformly converging Faber series (2.11) one chooses M such that $\|F - F_M\|$ is negligible. Then the Faber coefficients a_0, \dots, a_M of F_M are used to compute the CF approximation $p^{\mathcal{F}}(w) = b_0 + \dots + b_m w^m$ to $f_M := T^{-1}F_M$ on \bar{D} as described in Section 1 or, in more detail, in (Trefethen, 1981a). Finally, $P^{\mathcal{F}} := T p^{\mathcal{F}}$, i.e.

$$P^{\mathcal{F}}(z) = \sum_{k=0}^m b_k \phi_k(z).$$

In the case where $C = I := [-1, 1]$ and F is real-valued this CF-Faber method can be seen to be exactly the same as the method of asymptotic economization discussed by Lam (1972), D. Elliott (1973), Talbot (1976), and G. H. Elliott (1978). It differs only by a weight $\frac{1}{2}$ for the constant coefficient b_0 of q from Hollenhorst's version (Hollenhorst, 1976), but it differs to a greater extent from the method proposed by Darlington (1970) and Gutknecht & Trefethen (1982), which is more exact. However, $C = I$ is a very special case: Here, $\phi_k(\psi(w)) = 2T_k(\psi(w)) = w^k + w^{-k}$, so we get, cf. (3.3),

$$\begin{aligned} F_M(z) - P^{\mathcal{F}}(z) &= \sum_{j=0}^M b_j (w^j + w^{-j}) = q^+(w) + q^+ \left(\frac{1}{w} \right) \\ &= 2 \operatorname{Re} q^+(w). \end{aligned}$$

TABLE 1
Numerical results with the Faber-CF method on the domains (4.1), (4.2), and (4.3);
 $\varepsilon := \|F - P^{\varepsilon}f\|/\sigma - 1$

Γ	β	F	m	M	$\ F - F_m\ $	$\ F - F_M\ $	σ	$\ F - P^{\varepsilon}f\ $	ε
(4.1)	0.5	e^z	2	10	9.259(-2)	1.1(-9)	7.625(-2)	8.054(-2)	5.6(-2)
			4	12	2.333(-3)	4.1(-12)	2.072(-3)	2.083(-3)	5.5(-3)
			6	14	2.989(-5)	4.2(-14)	2.735(-5)	2.737(-5)	5.7(-4)
(4.1)	0.1	e^z	2	10	5.209(-2)	4.1(-11)	3.004(-2)	4.734(-2)	0.5761
			4	12	6.532(-4)	1.0(-13)	4.407(-4)	6.079(-4)	0.3796
			6	14	4.152(-6)	<(-13)	3.130(-6)	3.919(-6)	0.2520
(4.1)	0.5	$(1 + \frac{2}{3}z)^{\frac{1}{2}}$	4	17	2.209(-3)	1.5(-7)	1.535(-3)	1.547(-3)	7.9(-3)
			6	19	4.140(-4)	3.9(-8)	2.830(-4)	2.833(-4)	9.2(-4)
			8	21	8.771(-5)	1.0(-8)	5.930(-5)	5.931(-5)	1.1(-4)
(4.1)	0.1	$(1 + \frac{2}{3}z)^{\frac{1}{2}}$	4	17	6.166(-4)	8.9(-10)	3.483(-4)	4.887(-4)	0.4030
			6	19	6.037(-5)	1.3(-10)	3.662(-5)	4.656(-5)	0.2716
			8	21	6.885(-6)	2.0(-11)	4.403(-6)	5.206(-6)	0.1825
(4.2)	2.0	e^z	4	12	4.219(-1)	1.6(-6)	3.232(-1)	3.245(-1)	4.2(-3)
			6	14	3.555(-2)	3.0(-8)	2.876(-2)	2.879(-2)	1.0(-3)
			8	16	1.843(-3)	4.3(-10)	1.544(-3)	1.544(-3)	2.5(-4)
			10	18	6.395(-5)	5.3(-12)	5.499(-5)	5.499(-5)	5.8(-5)
(4.2)	1.2	e^z	4	12	2.974(-2)	2.0(-9)	2.342(-2)	2.603(-2)	1.1(-1)
			6	14	9.133(-4)	1.3(-11)	7.736(-4)	8.321(-4)	7.2(-2)
			8	16	1.742(-5)	<(-12)	1.516(-5)	1.593(-5)	5.1(-2)
			10	18	2.184(-7)	<(-12)	1.960(-7)	2.027(-7)	3.4(-2)
(4.2)	1.0	e^z	4	12	1.289(-2)	2.2(-10)	9.299(-3)	1.252(-2)	0.3462
			6	14	2.892(-4)	1.1(-12)	2.144(-4)	2.891(-4)	0.3483
			8	16	3.931(-6)	<(-13)	2.925(-6)	3.945(-6)	0.3487
			10	18	3.529(-8)	<(-13)	2.630(-8)	3.546(-8)	0.3483
(4.3)		e^z	2	8	1.456(-1)	5.4(-7)	1.111(-1)	1.657(-1)	0.4915
			4	10	4.529(-3)	3.0(-9)	3.269(-3)	4.924(-3)	0.5064
			6	12	6.471(-5)	1.1(-11)	4.611(-5)	6.859(-5)	0.4876
			8	14	5.415(-7)	<(-13)	3.799(-7)	5.671(-7)	0.4931
(4.3)	$(1 + 2z)^{-\frac{1}{2}}$		4	22	6.356(-2)	4.0(-5)	3.848(-2)	5.669(-2)	0.4732
			6	24	2.611(-2)	1.8(-5)	1.591(-2)	2.387(-2)	0.5003
			8	26	1.141(-2)	8.4(-6)	6.783(-3)	1.016(-2)	0.4980
			10	28	4.908(-3)	3.9(-6)	2.948(-3)	4.397(-3)	0.4915
			12	30	2.160(-3)	1.8(-6)	1.299(-3)	1.948(-3)	0.4998

Hence, the sum $\sum b_j(\phi_j - \phi^j)$ in (3.3) contributes exactly as much to the error as the error q^+ of the CF method for the disc does. In the general case the former error can be substantially bigger.

In Table 1 we summarize some of our *numerical experiments*. The domains treated are the *ellipse*

$$x^2 + \left(\frac{y}{\beta}\right)^2 \leq 1, \quad (4.1)$$

with semi-axes 1 and $\beta = 0.5$ or 0.1 , the *lemniscate*

$$|z^2 - 1| \leq \beta^2 \quad (4.2)$$

also called *oval of Cassini*, with $\beta = 2.0$, 1.2 , or 1.0 , and the *semi-disc*

$$x^2 + y^2 \leq 1, \quad x \geq 0. \quad (4.3)$$

The functions to be approximated are $F(z) = e^z$ (on all these domains), $F(z) = (1 + \frac{1}{2}z)^{\frac{1}{2}}$ (on the ellipse), and $F(z) = (1 + 2z)^{-\frac{1}{2}}$ (on the semi-disc). Table 1 lists the truncation error $\|F - F_M\|$, the singular value σ (which is a lower bound for the error of the best approximation of F_M), the error $\|F - P^{CF}\|$ of the Faber-CF approximation, its relative deviation $\varepsilon := \|F - P^{CF}\|/\sigma - 1$ from σ , and, for comparison, the error $\|F - F_m\|$ of the truncated Faber series.

The Faber polynomials ϕ_j were actually computed using the techniques discussed in (Ellacott, 1983a), although for the ellipse and the lemniscate their coefficients could be obtained analytically (Markushevich, 1967).

It is apparent from Table 1 that the Faber-CF approximation does extremely well on the fat ellipse ($\beta = 0.5$) and on the similar looking oval of Cassini with $\beta = 2.0$. Note that $\varepsilon + \|F - F_M\|/\sigma$ is an upper bound for the *relative* deviation of $\|F - P^{CF}\|$ from the error $E_m(F, C)$ of the best approximation. Here, this relative deviation is typically less than 10^{-3} already for $m = 4$ or 6 , and it becomes rapidly smaller as m increases. [However, this accuracy of the Faber-CF method is still far below the one obtained on a disc (Trefethen, 1981) and on an interval (Gutknecht & Trefethen, 1982).] On the flat ellipse ($\beta = 0.1$) ε decreases only slowly as m increases, but this must be expected from the fact mentioned at the end of Section 3 that on a degenerate ellipse ($\beta = 0.0$) both ε and $E_m(F, C)/\sigma - 1$ are typically close to 1. So, if ε is large, this does not necessarily imply that the method does not work in that particular example. On the flat ellipse ($\beta = 0.1$) and on Cassini's oval with $\beta = 1.2$ the method is still clearly better than truncation of the Faber series. Unfortunately, this is no more true on the semi-disc and on the lemniscate (5.2) with $\beta = 1$, whose interior is no more simply connected, the boundary having a double point at $z = 0$. On the semi-disc $1 + \varepsilon$ is always close to α defined in Theorem 3.2. (According to this theorem α is an asymptotic upper bound for $1 + \varepsilon$.) However, the true best approximation is often distinctly better than P^{CF} ; for example, according to G. H. Elliott (1978) $E_4(e^z, C) \doteq 3.7992(-3)$ (p. 91), $E_6(e^z, C) \doteq 5.070(-5)$ (p. 93), and $E_4((1 + 2z)^{-\frac{1}{2}}, C) \doteq 4.23412(-2)$ (p. 94) on the semi-disc. Of course, $\sigma - \|F - F_M\|$ is still a lower bound for $E_m(F, C)$, but from these numerical values we see that it is no longer very sharp. Similar results have been obtained for other functions. Hence, for the case of polynomial approximation on regions with non-analytic boundary the Faber-CF method does not appear to offer much improvement over simple truncation of the Faber series. On the other hand, the lower bound σ may still be useful.

Many of the ideas discussed here, however, have generalizations to rational approximation, and there the CF method turns out to offer spectacular improvements over the corresponding Padé-type approximant even when the boundary is not analytic (see Ellacott, 1983b).

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